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On the characteristic polynomial of homeomorphic images of a graph

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Abstract

The characteristic polynomial of a homeomorphic image $H(G)$ of an arbitrary graph G is expressed in terms of simpler characteristic polynomials. This is applied to obtain the characteristic polynomials of various known families of graphs, including the family of theta graphs, $H(K_{2,3})$. An improvement of the Chartrand–Harary theorem, which characterises outerplanar graphs and in which the theta graphs appear is presented here.

0. Introduction

Certain classes of graphs are identified as those not containing a particular subgraph S or a homeomorphic image $H(S)$ thereof. Many results, including Kuratowski's theorem on planar graphs, relate to subgraphs which are homeomorphs of certain graphs.

Graphs G and K are said to be homeomorphic images of each other and of a graph Γ if they can both be obtained from the graph Γ by inserting new vertices of degree 2 into its edges. We also say that G is a *homeomorph* of Γ and write $G = H(\Gamma)$; similarly for K. It is useful to be able to form the characteristic polynomial of a homeomorph $H(\Gamma)$ from those of simpler graphs. We present such a result which generalises theorems by Cvetković et al. including his derivation of the characteristic polynomial of the subdivision of a graph obtained by inserting exactly *one* vertex into *each* edge of G [3].

0.1. Preliminary results

Let A be the adjacency matrix of a graph G of order n and $\phi(G)$ be its characteristic polynomial $\det(\lambda I - A)$. If u, v are two vertices and e an edge of G then $G - e$ is defined

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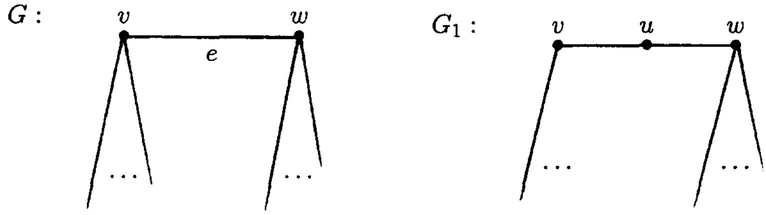


(i) (ii)

Fig. 1.

to be the graph obtained from G by deleting e, $G - v$ is defined to be obtained from G by deleting vertex v and all edges incident to it, and $G - u - v$ denotes the graph $(G - u) - v$. The vertex set and edge set of a graph G will be denoted by $\mathscr{V}(G)$ and $\mathscr{E}(G)$, respectively.

Schwenk defines the *coalescence* of the rooted graphs (W, w) and (V, v) obtained by identifying the vertices w and v so that the new root $v = w$ is a cut-vertex joining W to V [6]. A *path* P_k is a graph on vertex set $v_1, v_2, \ldots, v_k$ whose edge set is $\{(v_i, v_{i+1}): 1 \leqslant i \leqslant k - 1\}$. With a view to obtaining the characteristic polynomial of $H(G)$, also denoted by G_s, obtained, from G by replacing one of its edges by a path of length $s + 1$, we shall first consider obtaining the characteristic polynomial of the homeomorphic image G_1 obtained from G by replacing an edge e by a path P_3. Let the inserted vertex be u. The characteristic polynomials of G and G_1 are related as follows.

1. One vertex inserted: G_1

Theorem 1. *Let G be a graph, e an edge of G joining vertices v and w, and let G_1 be the homeomorphic image of G obtained by inserting a vertex u in edge e.*

Then

$$\phi(G_1) = \phi(G) - \phi(G - v) - \phi(G - w) + \phi(G - v - w) + (\lambda - 1)\phi(G - e). \tag{1}$$

The following notation and lemmas will be used in the proof of this theorem. Let the order of G be n and let $G - e$ have adjacency matrix $[a_{i,j}]$ for $2 \leqslant i \leqslant n + 1$, $2 \leqslant j \leqslant n + 1$. With reference to Fig. 1, $\mathscr{V}(G_1)$ is relabelled so that u, v, w receive subscripts 1, 2, 3, respectively. With respect to this labelling

$$\phi(G_1) = \left|\begin{array}{ccc|ccccc} \lambda & -1 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & \lambda & 0 & -a_{2,4} & -a_{2,5} & \cdots & \cdots & -a_{2,n+1} \\ -1 & 0 & \lambda & -a_{3,4} & -a_{3,5} & \cdots & \cdots & -a_{3,n+1} \\ \hline 0 & -a_{4,2} & -a_{4,3} & & & & & \\ \vdots & \vdots & \vdots & & & B & & \\ 0 & -a_{n+1,2} & -a_{n+1,3} & & & & & \end{array}\right|. \tag{2}$$

Note that $\det B = \phi(G - v - w) = \det(\lambda I - C)$, where C is the adjacency matrix of $G - v - w$.

Thus,

$$\phi(G) = \left|\begin{array}{cc|ccccc} \lambda & -1 & -a_{2,4} & -a_{2,5} & \cdots & \cdots & -a_{2,n+1} \\ -1 & \lambda & -a_{3,4} & -a_{3,5} & \cdots & \cdots & -a_{3,n+1} \\ \hline -a_{4,2} & -a_{4,3} & & & & & \\ \vdots & \vdots & & & B & & \\ -a_{n+1,2} & -a_{n+1,3} & & & & & \end{array}\right|, \tag{3}$$

so that the entries of $\phi(G - e)$ and $\phi(G)$ differ in that the entries -1 in the first row and first column of $\phi(G)$ are replaced by 0.

Let ${}^2\Delta^3$ denote the following determinant of order $(n - 1)$.

$${}^2\Delta^3 = \left|\begin{array}{c|ccccc} 0 & -a_{2,4} & -a_{2,5} & \cdots & \cdots & -a_{2,n+1} \\ \hline -a_{4,3} & & & & & \\ -a_{5,3} & & & B & & \\ \vdots & & & & & \\ -a_{n+1,3} & & & & & \end{array}\right|. \tag{4}$$

Lemma 1.1. *The characteristic polynomial of* G_1 *defined in* (2) *is*

$$\phi(G_1) = \lambda\phi(G - e) - \phi(G - v) - \phi(G - w) + 2({}^2\Delta^3). \tag{5}$$

Proof. Expanding determinant $\phi(G_1)$ by the first row it is seen that it is the sum of three terms:

$$\phi(G_1) = \lambda\phi(G - e) + (-1)\Upsilon_1 + (-1)\Upsilon_2,$$

where $\phi(G - e)$, Υ_1 and Υ_2, are the respective cofactors of the three non-zero entries of the first row in that order.

The determinants Υ_1 and Υ_2 in turn may be expanded as follows:

$$\Upsilon_1 = \phi(G - v) - {}^2\Delta^3$$

and

$$\Upsilon_2 = \phi(G - w) - {}^2\Delta^3.$$

The result follows. □

We now consider the difference $\phi(G) - \phi(G - e)$ of the two nth-order determinants.

Lemma 1.2. *If* $B = \phi(G - v - w)$, *then*

$$\phi(G) - \phi(G - e) = -B + 2({}^2\Delta^3). \tag{6}$$

Proof. Expanding determinant $\phi(G)$ by the first row

$$\phi(G) = \lambda\phi(G - v) + T' + (-a_{2,4})T_{2,4} + \cdots + (-1)^{n+1}(-a_{2,n+1})T_{2,n+1},$$

where T' is the minor of the entry (-1) and $(-1)^i T_{2,i}$ is the minor of entry $(-a_{2,i})$ for $4 \leqslant i \leqslant n + 1$ in the determinant $\phi(G)$.

Similarly,

$$\phi(G - e) = \lambda\phi(G - v) + (-a_{2,4})S_{2,4} + \cdots + (-1)^{n+1}(-a_{2,n+1})S_{2,n+1},$$

where $(-1)^i S_{2,i}$ is the minor of entry $(-a_{2,i})$ for $4 \leqslant i \leqslant n + 1$ in the determinant $\phi(G - e)$.

Subtracting, we get

$$\phi(G) - \phi(G - e) = T' + (-a_{2,4})(T_{2,4} - S_{2,4}) + \cdots + (-1)^n(-a_{2,n+1})(T_{2,n+1} - S_{2,n+1}).$$

Now, $T_{2,i} - S_{2,i} = {}^2\Delta^3(i)$ which is the minor of $(-a_{2,i})$ in the determinant ${}^2\Delta^3$.

Also, T' is the sum of two determinants obtained by expanding T' by the first row:

$$T' = -B + {}^2\Delta^3.$$

The result follows. □

Proof of Theorem 1. The result follows from Lemmas 1.1 and 1.2 by eliminating $2({}^2\Delta^3)$ from Eqs. 5 and 6. □

2. Tailed structures

Definition. Let ${}^sW(w)$ denote the *coalescence* of the rooted graphs (W, w) and path (P_{s+1}, v) so that the end vertex v of P_{s+1} identifies with the vertex w of graph W. ${}^sW(w)$ may be looked at as the graph W with a tail of length s attached at vertex w of W [6].

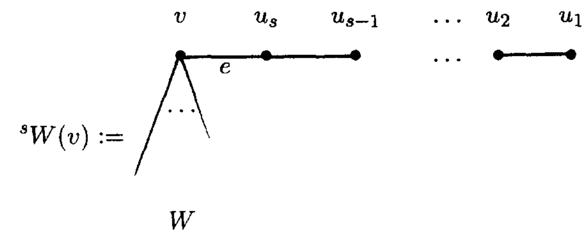


Fig. 2.

Lemma 2.1. *For $s \geqslant 1$,*

$$\phi({}^sW(v)) = \lambda\phi({}^{s-1}W(v)) - \phi({}^{s-2}W(v)), \tag{7}$$

where we set ${}^{-1}W(v) := W - v$ and ${}^0W(v) := W$. (Fig. 2)

Proof. We prove this directly by evaluating

$$\phi({}^sW(v)) = \det(-{}^sW(v) + \lambda I), \quad s \geqslant 1.$$

Let the vertices of the set $\mathscr{V}({}^sW(v))$ be relabelled so that u_1 and u_2 are represented by the first and second rows of the adjacency matrix of ${}^sW(v)$. Let $[b_{i,j}]$ be the adjacency matrix of ${}^{s-1}W(v)$. Then, for $s \geqslant 1$,

$$\phi({}^sW(v)) = \left|\begin{array}{cc|ccccc} \lambda & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & \lambda & -b_{2,3} & -b_{2,4} & \dots & \dots & -b_{2,n} \\ \hline 0 & -b_{3,2} & & & & & \\ 0 & -b_{4,2} & & & & & \\ 0 & -b_{5,2} & & & T & & \\ \vdots & \vdots & & & & & \\ 0 & -b_{n,2} & & & & & \end{array}\right|, \tag{8}$$

where det T is the characteristic polynomial of ${}^{s-2}W(v)$.

Expanding by the first row,

$$\phi({}^sW(v)) = \lambda\phi({}^{s-1}W(v)) + (-1)\det(T),$$

from which the result follows. $\square$

Note that a different proof of this result is given in [6,7].

Lemma 2.2. *The characteristic polynomials of ${}^sW(v)$ for $s = 0, 1, 2, \dots$, are given by the generating function*

$$S(x) = \sum_{s=0}^{\infty} \phi({}^sW(v))x^s, \tag{9}$$

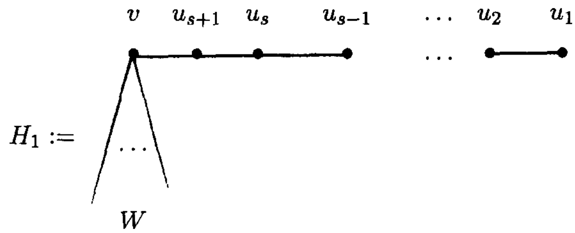


Fig. 3.

where $S(x)$ satisfies

$$(1 - \lambda x + x^2)S(x) = \phi(W) - x\phi(W - v). \tag{10}$$

Proof. The recurrence relation from Lemma 2.1,

$$\phi({}^sW(v)) = \lambda\phi({}^{s-1}W(v)) - \phi({}^{s-2}W(v)),$$

multiplied by x^s, yields

$$\phi({}^sW(v))x^s = \lambda x\phi({}^{s-1}W(v))x^{s-1} - x^2\phi({}^{s-2}W(v))x^{s-2}.$$

Summing over s,

$$\sum_{s=2}^{\infty} \phi({}^sW(v))x^s = \lambda x \sum_{s=2}^{\infty} \phi({}^{s-1}W(v))x^{s-1} - x^2 \sum_{s=2}^{\infty} \phi({}^{s-2}W(v))x^{s-2}$$

is obtained.

Then

$$S(x) - \phi({}^1W(u))x - \phi(W) = \lambda x(S(x) - \phi(W)) - x^2S(x),$$

that is,

$$S(x)(1 - \lambda x + x^2) = \phi(W)(1 - \lambda x) + x\phi({}^1W(v)),$$

where $\phi({}^1W(v)) = \lambda\phi(W) - \phi(W - v)$, by Lemma 2.1.

Hence,

$$S(x)(1 - \lambda x + x^2) = \phi(W) - x\phi(W - v). \qquad \square$$

Lemma 2.3. *Let e be the edge vu_s in the tailed structure $H = {}^sW(v)$ and let H_1 be obtained from H by inserting vertex u_{s+1} into edge e. If H_1 is relabelled so that v and u_s correspond to the 2nd and 3rd rows in the adjacency matrix of H_1, then ${}^2\Delta^3 = 0$.*

Proof. Referring to Fig.3 let the order of W be p so that the order of H_1 is $p+s+1$. A labelling for H_1 is chosen so that, in its adjacency matrix, vertices u_{s+1}, v, u_s, $u_{s-1}, \dots,$

u_2, u_1, correspond to the 1st, 2nd, 3rd, 4th, ..., $(s+2)$th rows, respectively, and the vertices of $W - v$ correspond to the remaining rows, $s+3, s+4, \dots, s+p+1$. Then

$$
{}^2\Delta^3 = \left|\begin{array}{c:cccccc|cccc}
0 & 0 & 0 & 0 & \dots & 0 & 0 & -a_{s+3,2} & -a_{s+4,2} & \dots & -a_{s+p+1,2} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
-1_{4,3} & \lambda & -1_{4,5} & 0 & \dots & 0 & 0 & & & & \\
0 & -1_{5,4} & \lambda & -1_{5,6} & \dots & 0 & 0 & & & & \\
\vdots & & & & & \vdots & \vdots & & & O & \\
0 & & & & & -1 & \lambda & & & & \\
\hline
0_{s+3,3} & & & & & & & & & & \\
\vdots & & & O & & & & & W-v & & \\
0_{s+p+1,3} & & & & & & & & & &
\end{array}\right| . \tag{11}
$$

Clearly, this determinant is of the form

$$
{}^2\Delta^3 = \left|\begin{array}{c|c} A & B \\ \hline O & W-v \end{array}\right|,
$$

where A is a square matrix of order s with a row having all entries equal to zero, and O is the zero matrix of order $p-1$ by s.

Thus,

$$
\begin{aligned}
{}^2\Delta^3 &= \det(A)\det(W-v) \\
&= 0
\end{aligned}
$$

Corollary 2.1. *Let e be the edge vw in the tail of the tailed structure G and let G_1 be obtained from G by inserting vertex u into edge e. Then*

$$\phi(G_1) = \lambda\phi(G-e) - \phi(G-v) - \phi(G-w), \tag{12}$$

and

$$\phi(G) = \phi(G-e) - \phi(G-v-w). \tag{13}$$

Proof. The results follow from Lemmas 1.1–2.3. □

2.1. Characteristic polynomials of particular tailed structures

2.1.1. Paths

(i) That

$$\phi(P_{n+s}) = \phi(P_n)\phi(P_s) - \phi(P_{n-1})\phi(P_{s-1}) \tag{14}$$

follows from (13) by deleting edge $v_n v_{n+1}$ of path P_{n+s} whose vertices are labelled $v_1, v_2, \ldots, v_{n+s}$ in order.

(ii) That

$$\phi(P_n) = \lambda\phi(P_{n-1}) - \phi(P_{n-2}) \tag{15}$$

follows from Lemma 2.1 as well as from (14) by letting $m = n - 1,\ s = 1$.

Using $\phi(P_0) = 1$ and $\phi(P_1) = \lambda$ as boundary conditions, [4], for this linear recursion the solution

$$\phi(P_n) = A\alpha^n + B\beta^n \tag{16}$$

is obtained by standard methods [1], where A and B determined by the boundary conditions, are found to be

$$A = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2\sqrt{\lambda^2 - 4}}, \qquad B = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2\sqrt{\lambda^2 - 4}},$$

and where α, β are the roots of the auxiliary equation

$$t^2 - \lambda t + 1 = 0.$$

Hence, the characteristic polynomial is

$$\phi(P_n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = 1 + \frac{\alpha}{\beta} + \left(\frac{\alpha}{\beta}\right)^2 + \cdots + \left(\frac{\alpha}{\beta}\right)^n, \tag{17}$$

where $\alpha/\beta = (\lambda + \sqrt{\lambda^2 - 4})^2/4$. Thus, $\phi(P_n) = 0$ has n roots given by

$$\frac{\alpha}{\beta} = \mathrm{e}^{\mathrm{i}2\pi k/(n+1)}, \quad k = 1, 2, \ldots, n.$$

The characteristic roots are therefore

$$\lambda = 2\cos\frac{\pi k}{n+1}, \quad k = 1, 2, \ldots, n. \tag{18}$$

as is well known.

(iii) Also, if

$$P(x) = \sum_{n=0}^{\infty} \phi(P_n)x^n,$$

then

$$P(x)(1 - \lambda x + x^2) = 1. \tag{19}$$

It is noted that this result is quoted in [4].

If W is the single vertex P_1 and ${}^sW(v) = P_{s+1}$, since

$$S(x) = P_1 + P_2x + P_3x^2 + \cdots$$

then, using Lemma 2.2 (19) follows.

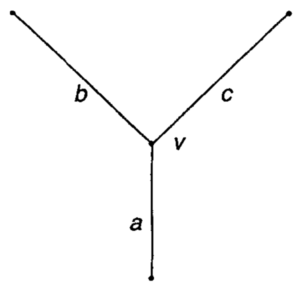


Fig. 4.

2.1.2. Stars

(i) Following Collatz and Sinogowitz [2], we define a star $S_{a_1,a_2,\dots,a_n}(v)$ to be the coalescence of n paths $P_{a_1+1}, P_{a_2+1}, \dots, P_{a_n+1}$ so that the n end-vertices, one from each path, are identified with each other in vertex v. We shall need the three-star $S_{a,b,c}(v)$ which is the coalescence of three paths P_{a+1}, P_{b+1} and P_{c+1} (see Fig. 4).

$$S_{a,b,c}(v) = {}^{a}V_{bc}(v)$$

The characteristic polynomial of the star satisfies

$$\phi(S_{a,b,c}(v)) = \phi(P_{a+c+1})\phi(P_b) - \phi(P_a)\phi(P_{b-1})\phi(P_c). \tag{20}$$

Proof. The result follows from (13) by deleting the edge in P_{b+1} adjacent to v. □

(ii) The characteristic polynomials of stars $S_{r,s,t}$, denoted by ${}^{r}V$, when s and t remain unchanged, for $r = 0, 1, 2, \dots$, are given by the generating function

$$\Omega(x) = \sum_{r=0}^{\infty} \phi({}^{r}V)x^r, \tag{21}$$

where $\Omega(x)$ satisfies the equation

$$\Omega(x)(1 - \lambda x + x^2) = \phi(P_{s+t+1}) - x\phi(P_s)\phi(P_t). \tag{22}$$

The proof follows from Lemma 2.2 for tailed structures.

2.1.3. Lollipops

Following Thomason we call $L_1 = {}^{s}C_n$ obtained, by coalescing path P_{s+1} of arbitrary length with a circuit C_n, with an end point of the path and a vertex of the circuit identified, a lollipop (see Fig. 5).

$$L_1 = {}^{s}C_n$$

(i) The characteristic polynomial of L_1 satisfies (Fig. 6).

$$\phi(L_1) = \phi(C_n)\phi(P_s) - \phi(P_{n-1})\phi(P_{s-1}). \tag{23}$$

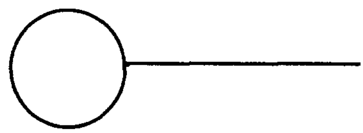


Fig. 5.

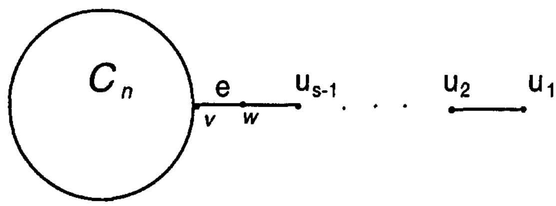


Fig. 6.

Proof. The proof follows from (13). □

(ii) The characteristic polynomials of lollipops $\phi({}^rC_n)$ denoted by ${}^rC$, when n remains unchanged, for $r = 0, 1, 2, \ldots$, are given by the generating function

$$C(x) = \sum_{r=0}^{\infty} \phi({}^rC)x^r, \tag{24}$$

where by Lemma 2.2, $C(x)$ satisfies

$$C(x)(1 - \lambda x + x^2) = \phi(C_n) - x\phi(P_{n-1}). \tag{25}$$

It is noted that Cvetković et al. gave a table for the characteristic polynomials of ${}^tC_5$, ${}^tC_6$ and ${}^tC_7$ for $1 \leqslant t \leqslant 7$ [3].

Thus, the characteristic polynomials of the above tailed structures may all be expressed in terms of those of paths and circuits.

3. The general homeomorph

Theorem 2. *Let G_s denote the graph G with s vertices $u_1, u_2, \ldots, u_s$ inserted into an edge e joining vertices v and w of G. Then for $s \geqslant 1$ (see Fig. 7),*

$$\begin{aligned}\phi(G_s) = {} & \phi(G) + \phi({}^s(G-e)(w)) - \phi(G_s - v - u_s)\\ & - \phi(G-e) + \phi(G - v - w).\end{aligned} \tag{26}$$

Proof. Let Π_s be the proposition

$$\phi(G_s) = \phi(G) + \phi({}^s(G-e)(w)) - \phi(G_s - v - u_s) - \phi(G-e) + \phi(G-v-w).$$

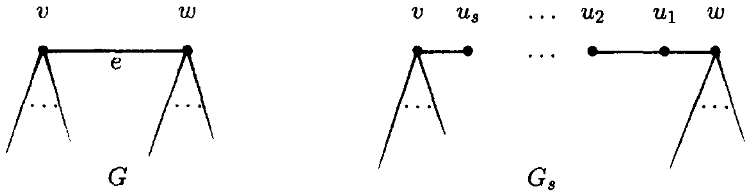


Fig. 7.

Π_1 is true as

$$\begin{aligned}\text{R.H.S.} &= \phi(G)+\phi({}^1(G-e)(w))-\phi(G_1-v-u_1)-\phi(G-e)+\phi(G-v-w)\\ &= \phi(G)+\lambda\phi(G-e)-\phi(G-w)-\phi(G-v)-\phi(G-e)+\phi(G-v-w)\\ &= \phi(G_1)\ldots\text{from Theorem 1.}\end{aligned}$$

Assuming Π_s, using Theorem 1 and taking u_{s+1} as the inserted vertex u in edge e joining v to u_s,

$$\begin{aligned}\phi(G_{s+1}) &= \phi(G_s) - \phi(G_s - v) - \phi(G_s - u_s) + \phi(G_s - v - u_s)\\ &\quad + (\lambda - 1)\phi(G_s - e)\\ &= \phi(G_s) - \phi(G_{s+1} - v - u_{s+1}) - \phi(G_{s+1} - u_s - u_{s+1})\\ &\quad + \phi(G_{s+1} - v - u_s - u_{s+1}) + (\lambda - 1)\phi(G_{s+1} - u_{s+1}).\end{aligned}$$

Recalling that ${}^s(G-e)(w) = G_{s+1} - u_{s+1} = G_s - e$ and using the hypothesis that Π_s is true,

$$\begin{aligned}\phi(G_{s+1}) &= \phi(G) + \phi(G_{s+1} - u_{s+1}) - \phi(G_{s+1} - v - u_s - u_{s+1}) - \phi(G - e)\\ &\quad + \phi(G - v - w) - \phi(G_{s+1} - v - u_{s+1}) - \phi(G_{s+1} - u_s - u_{s+1})\\ &\quad + \phi(G_{s+1} - v - u_s - u_{s+1}) + (\lambda - 1)\phi(G_{s+1} - u_{s+1}).\end{aligned}$$

Since $\lambda\phi(G_{s+1} - u_{s+1}) - \phi(G_{s+1} - u_s - u_{s+1}) = \phi({}^{s+1}(G-e)(w))$, the truth of Π_{s+1}, is deduced. The result follows by induction on s. □

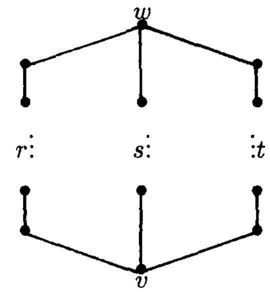
Corollary 3.1. *The characteristic polynomial of cycle C_n is*

$$\phi(C_n) = \phi(C_3) + \phi(P_n) - \phi(P_{n-2}) - \phi(P_3) + \phi(P_1) \tag{27}$$

Proof. The proof follows from Eq. (26) of Theorem 2. □

Corollary 3.2. *If e is an isthmus of G joining vertices v and w of G so that $G - e$ is the disjoint union of the graphs V and W then*

$$\begin{aligned}\phi(G_s) &= \phi(G) + \phi(V)\phi({}^sW) - \phi(V)\phi(W)\\ &\quad +\phi(V - v)\phi(W - w) - \phi(V - v)\phi({}^{s-1}W).\end{aligned} \tag{28}$$



$K_{2,3}(r,s,t)$

Fig. 8.

Proof. The proof follows from (26) as the characteristic polynomial of a disconnected graph is the product of the characteristic polynomials of its components. □

4. Applications

The above theorems are now applied to find the characteristic polynomial of the theta graphs, viz., the homeomorphic images of $K_{2,3}$.

4.1. The homeomorphic image of $K_{2,3}$

Let $K_{2,3}(r,s,t)$ denote the homeomorphic images of $K_{2,3}$ such that each of the three paths joining the vertices of degree 3 have r,s,t, vertices of degree 2 inserted in them, respectively (see Fig. 8),

Notation. $K_{2,3}(r,s,t)$ will be denoted by $K_{2,3}(r)$ when the positive integers s and t remain unchanged. The tailed structure ${}^aV_{bc}(v)$, is the star $S_{a,b,c}$, where the tail of length a is incident to vertex v.

Lemma 4.1. *The characteristic polynomial of $K_{2,3}(r)$ is given by*

$$\begin{aligned}\phi(K_{2,3}(r)) = \phi(K_{2,3}(r-1)) &- (\phi({}^{r-1}V_{st}(v)) - \phi({}^{r-2}V_{st}(v)))\\ &+ (\phi({}^{r}C_{s+t+2}(v)) - \phi({}^{r-1}C_{s+t+2}(v)))\end{aligned} \tag{29}$$

Proof. Theorem 1 is now applied to express $\phi(K_{2,3}(r))$ in terms of $\phi(K_{2,3}(r-1))$. If e is the edge adjacent to w on the wv path of length $r+1$ then

$$\begin{aligned}\phi(K_{2,3}(r)) = \phi(K_{2,3}(r-1)) &- \phi({}^{r-1}V_{st}(v)) - \phi({}^{r-2}C_{s+t+2}(v))\\ &+ \phi({}^{r-2}V_{st}(v)) + (\lambda - 1)\phi({}^{r-1}C_{s+t+2}(v)).\end{aligned}$$

Since by (7),

$$\lambda\phi({}^{r-1}C_{s+t+2}) - \phi({}^{r-2}C_{s+t+2}) = \phi({}^{r}C_{s+t+2}),$$

the result follows. □

Theorem 3. *If the generating function of the characteristic polynomials of the* $K_{2,3}(r)$ *is*

$$K(x) = \phi(K_{2,3}(0)) + \phi(K_{2,3}(1))x + \phi(K_{2,3}(2))x^2 + \cdots,$$

then

$$K(x) = \phi(K_{2,3}(0)) + \phi(K_{2,3}(1))\left(\frac{x}{1-x}\right) + \phi(C_{s+t+2})\left(\frac{1}{1-\lambda x+x^2} - 1 - \frac{\lambda x}{1-x}\right)$$
$$+\phi(P_{s+t+1})\left(\frac{-2x}{1-\lambda x+x^2} + \frac{2x}{1-x}\right) + \phi(P_s)\phi(P_t)\left(\frac{x^2}{1-\lambda x+x^2}\right). \quad (30)$$

Proof. Multiplying (29) by x^r

$$\phi(K_{2,3}(r))x^r = x\phi(K_{2,3}(r-1))x^{r-1} - x\phi({}^{r-1}V_{st}(v))x^{r-1} + x^2\phi({}^{r-2}V_{st}(v))x^{r-2}$$
$$+ \phi({}^{r}C_{s+t+2})x^r - x\phi({}^{r-1}C_{s+t+2})x^{r-1}. \quad (31)$$

Summing over r, for $r \geqslant 2$, and using the generating functions $\Omega(x)$ and $C(x)$ for the stars $\phi({}^{r}V_{st})$ and the lollipops $\phi({}^{r}C_{s+t+2})$, respectively, from Eqs. (22) and (25) of Section 2.1,

$$K(x) - \phi((K_{2,3}(0)) - \phi((K_{2,3}(1))x$$
$$= x(K(x) - \phi(K_{2,3}(0))) - x(\Omega(x) - \phi({}^{0}Vst(v))) + x^2\Omega(x)$$
$$+ C(x) - \phi(C_{s+t+2}) - x\phi({}^{1}C_{s+t+2}) - x(C(x) - \phi(C_{s+t+2})).$$

By Lemma 2.1

$$\phi({}^{1}C_{s+t+2}) = \lambda\phi(C_{s+t+2}) - \phi(P_{s+t+1}).$$

Thus,

$$(1-x)K(x) = (1-x)\phi(K_{2,3}(0)) + x\phi(K_{2,3}(1)) + (1-x)C(x)$$
$$+ (-\lambda x - (1-x))\phi(C_{s+t+2}) + x(1-x)\Omega(x) + 2x\phi(P_{s+t+1}).$$

The result now follows. □

The use of this theorem is demonstrated to find the characteristic polynomial of the subdivision of $K_{2,3}$ which is $K_{2,3}(3,3,3)$ by the above notation. The coefficient of x^3 in the expansion of $K(x)$ is the required polynomial.

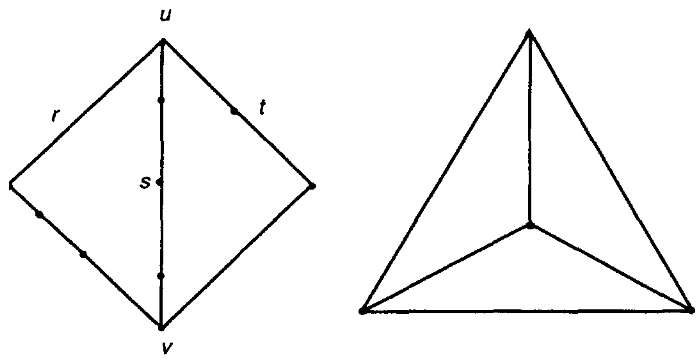


Fig. 9.

Thus,

$$\phi(K_{2,3}(3,3,3)) = \phi(K_{2,3}(1,3,3)) + \phi(C_8)(\lambda^3 - 3\lambda) + \phi(P_7)(-2\lambda^2 + 4) + \lambda\phi(P_3)\phi(P_3).$$

The well-known formula for the subdivision of a graph, [3, p. 63] may be used to calculate $\phi(K_{2,3}(1,3,3))$ since the graph is the subdivision of $K_4 - e$.

Hence,

$$\begin{aligned}
&\phi(K_{2,3}(3,3,3)) \\
&\quad = \lambda^9 - 10\lambda^7 + 32\lambda^5 - 40\lambda^3 + 16\lambda + (\lambda^3 - 3\lambda)(\lambda^8 - 8\lambda^6 + 20\lambda^4 - 16\lambda^2) \\
&\qquad + (-2\lambda^2 + 4)(\lambda^7 - 6\lambda^5 + 10\lambda^3 - 4\lambda) + \lambda(\lambda^3 - 2\lambda)^2 \\
&\quad = \lambda^{11} - 12\lambda^9 + 51\lambda^7 - 92\lambda^5 + 60\lambda^3.
\end{aligned}$$

4.2. *Minimally non-outerplanar graphs*

The theorem of Chartrand and Harary on outerplanar graphs states that these are characterised as those graphs *not* containing $K_{2,3}$ or K_4 or their homeomorphic images as subgraphs, (except $K_4 - e$) [5,9].

Let F denote K_4 or $H(K_{2,3})$ (Fig. 9).

Since $H(K_4)$ contains $H(K_{2,3})$ as a subgraph, the Chartrand–Harary theorem is more clearly stated as follows:

Theorem 4. *A graph G is not outerplanar if and only if G contains K_4, or $H(K_{2,3})$ as a subgraph.*

Definition. A graph G is defined to be *minimally non-outerplanar* if G is *not* outerplanar but $\forall\ e \in \mathscr{E}(G)$, $G - e$ is outerplanar [5].

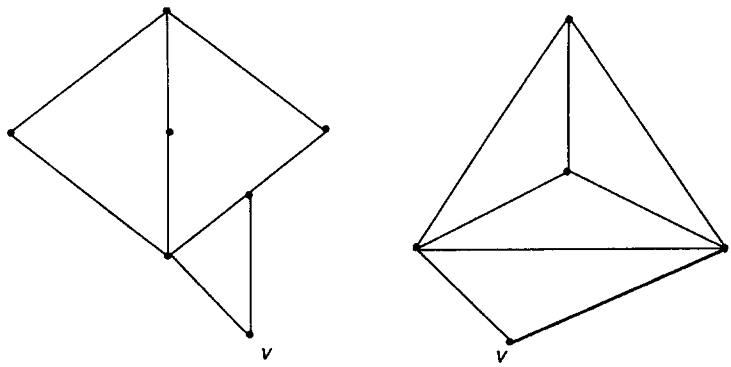


Fig. 10.

Clearly, K_4 and $K_{2,3}$ are both minimally non-outerplanar.

Mitchem has derived a similar result in terms of vertex deletions [8].

Theorem 5. *A connected graph G is minimally non-outerplanar if and only if G is K_4, or $H(K_{2,3})$.*

Proof. Assume G is minimally non-outerplanar. Then G contains F, and $\forall\ e \in \mathscr{E}(G)$, $G - e$ does not.

Case (i): Suppose G is F together with other vertices and edges. The edge set $\mathscr{E}(G \backslash \mathscr{E}(F))$ is not empty. If $e \in \mathscr{E}(G \backslash \mathscr{E}(F))$ is deleted, $H(K_{2,3})$ or $H(K_4)$ is still a subgraph of G and thus $G - e$ is non-outerplanar. Hence, this type of graph is not admissible as a minimally outerplanar graph. Thus, G has the same vertex set as F (Fig. 10).

Case (ii): Suppose G has the same vertex set as F but with extra edges joining vertices of F. As K_4 is complete, it is now excluded from F. It is clear that when e, one of these edges, is deleted, G would still contain $H(K_{2,3})$. This contradicts G as a minimally non-outerplanar graph each of whose edge-deleted subgraphs is outerplanar. Hence, this type of graph is not admissible as a minimally outerplanar graph (Fig. 11).

Thus, the graphs which are minimally non-outerplanar are F with no extra vertices or edges.

Conversely, let $G = F$. Then G is non-outerplanar.

Now it is clear that each edge-deleted subgraph of G is outerplanar. Hence, G is minimally non-outerplanar. $\square$

Corollary 4.1. *If G is not K_4, then G is minimally non-outerplanar if and only if its characteristic polynomial is a coefficient of x^r, $r \geqslant 1$, in the generating function $K(x)$.*

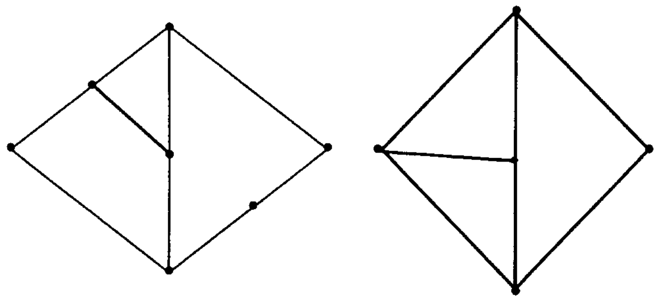


Fig. 11.

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